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TWO-DIMENSIONAL DYNAMIC PROCESSES IN ANISOTROPIC MEDIA *

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Dynamic processes in layers with an arbitrary mesh of the principal directions of anisotropy are investigated. A case corresponding to a curvilinear, homogeneous anisotropic medium is studied. Solutions are given to a number of concrete problems.

1. When the medium is anisotropic, the process is stationary and the basic dynamic law is linear then the equations of motion and heat transfer in the phenomena of filtration of liquids (D'Arcy's law) and heat conductivity (Fourier's law), have the form

$$\mathbf{v} = \mathbf{T}\nabla\varphi, \quad \nabla\mathbf{v} = 0 \tag{1.1}$$

Here v denotes, respectively, the rate of filtration and the thermal flux; $\varphi = -(P + \gamma z) / \mu$ in the case of filtration (P is the pressure of fluid, γ is acceleration due to gravity and μ is the viscosity of fluid), and $\varphi = -t$ in the case of heat conduction (t is temperature). Under the same conditions the magnetic and electric fields can be described, by virtue of the Maxwell's equations and the Ohm's law, by the relations

$$\mathbf{v} = \nabla\varphi, \quad \nabla\mathbf{T}\mathbf{v} = 0 \tag{1.2}$$

Here v denotes, respectively, the magnetic field intensity and current density and $\varphi = -V$ where V is the magnetic and electric potential. In equations (1.1) and (1.2) **T** is a second rank tensor, the coefficients of which depend on the coordinates and characterize, respectively, the inhomogeneity, magnetic permeability and specific resistance of the medium, and its anisotropy.

When the process is two-dimensional equations (1.1) have the form

$$v_i = k_{i1}\partial\varphi / \partial x_1 + k_{i2}\partial\varphi / \partial x_2 = (-1)^j H^{-1}\partial\psi / \partial x_j \tag{1.3}$$
$$i = 1,\ j = 2; \quad i = 2,\ j = 1$$

Here x_i denote the Cartesian coordinates of the (z)-plane along which the process develops, or, onto which the process is mapped conformally in a layer distributed along a curvilinear surface, v_i are the projections of the vector v on the x_i-axis, H denotes the law of variation in the thickness of the layer and is, generally, dependent on the coordinates /1/, and ψ is the stream function. Equations (1.2) are analogous in the case of a two-dimensional process

$$v_i = \partial\varphi / \partial x_i = (-1)^j(k_{i1}\partial\psi / \partial x_1 + k_{i2}\partial\psi / \partial x_2) \tag{1.4}$$
$$i = 1,\ j = 2;\ i = 2,\ j = 1$$

Let us show the nature of the anisotropy of the medium described by tensor **T**, in the two-dimensional case. We shall regard the motion at a given point in the direction s along the vector v as one-dimensional and will use the coefficient k_s to describe the medium in this direction. When the process obeys a linear law, the motion along s is described by the equation

$$v = k_s(\cos\alpha_1\partial\varphi / \partial x_1 + \cos\alpha_2\partial\varphi / \partial x_2) \tag{1.5}$$

where $\cos\alpha_1$ and $\cos\alpha_2$ are the direction cosines.

Let us introduce a vector of length $\sqrt{k_s}$ with the origin at a specified point and the coordinates $X_i = \sqrt{k_s}\cos\alpha_i$, and a tensor **F** with components f_{ij} such that $\nabla\varphi = \mathbf{F}\mathbf{v}$. Then we can write the equations (1.4) in the form

$$f_{11}X_1^2 + (f_{12} + f_{21})\, X_1X_2 + f_{22}X_2^2 = 1 \tag{1.6}$$

Since $k_s \neq \infty$, it follows from (1.5) that $\sqrt{k_s}$ varies with s according to an elliptical law, and this corresponds to the tensor type of the equations (1.3). Thus (1.3), and in the same manner (1.4), describe the dynamic processes in a medium possessing a particular type of anisotropy. Experimental confirmation of such a character of the anisotropy can be found in e.g. /2/.

Let us introduce an orthogonal coordinate system p_i in the (z)-plane, directed along the principal axes of the ellipses described by (1.6). Then using these variables, we encounter two cases: 1) $f_{12} = f_{21} = 0$ or $k_{12} = k_{21} = 0$, and 2) $f_{12} = -f_{21}$ or $k_{12} = -k_{21}$.

When $k_{12} = k_{21}$, then the square of the arc element dS^2 in the (z)-plane is given by the following expression in terms of the curvilinear coordinates p_i:

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$$dS^2 = H_1^2 dp_1^2 + H_2^2 dp_2^2 \tag{1.7}$$

where H_1 and H_2 denote the Lamé coefficients.

Equations (1.3) in these coordinates become

$$v_i = k_i H_i^{-1} \partial\varphi / \partial p_i = (-1)^j (H_j H)^{-1} \partial\psi / \partial p_j \qquad i = 1,\ j = 2; \quad i = 2,\ j = 1 \tag{1.8}$$

Here v_i denote the projections of the vector v on the p_i-coordinate axes, k_i are the coefficients describing the medium in the directions p_i, and H_i are given by (1.7). When $k_{12} = k_{21}$, the coordinates p_i are called the principal directions of the anisotropy of the medium, and equations (1.8) become referred to the principal directions. An analogous coordinate system was used in the problems of diffusion by filtration /3/.

When $k_{12} = k_{21}$, equations (1.4) referred to the principal directions of the anisotropy of the medium have the form

$$v_i = H_i^{-1} \partial\varphi / \partial p_i = (-1)^j (k_i H_j H)^{-1} \partial\psi / \partial p_j \qquad i = 1,\ j = 2; \quad i = 2,\ j = 1 \tag{1.9}$$

When $k_{12} = -k_{21}$, equations (1.3) and (1.4) written in terms of the variables p_i are not reduced to the form (1.8) and (1.9). The principal directions of the anisotropy are, in this case, no longer orthogonal, therefore it is expedient to use non-orthogonal coordinates when writing the equations in canonical form. Let us write the equations (1.8) in the form

$$\sqrt{K_1/K_2}\, \partial\varphi / \partial p_i = (-1)^j \partial\psi / (\sqrt{K_1 K_2}\, \partial p_j)$$
$$K_i = k_i H H_j / H_i, \quad i = 1,\ j = 2; \ i = 2,\ j = 1$$

We introduce the auxilliary variable ξ_i which satisfy the Beltrami equations

$$\sqrt{K_i/K_j}\, \partial\xi_1 / \partial p_i = (-1)^j \partial\xi_2 / \partial p_j \qquad i = 1,\ j = 2; \ i = 2,\ j = 1 \tag{1.10}$$

The variables ξ_i and ξ_2 represent quasiconformal transformation of the (z)-plane into the (ζ)-plane with variables ξ_1, and equations (1.8), (1.9) written in these variables become

$$\partial\varphi / \partial\xi_i = (-1)^j \partial\psi / (H\sqrt{k_1 k_2}\, \partial\xi_j) \qquad i = 1,\ j = 2; \quad i = 2,\ j = 1 \tag{1.11}$$

Solutions of these equations can be written in terms of the P-analytic functions.

Equations (1.11) represent a canonical form of the differential equations (1.2) (see e.g. /4/). Their physical meaning is, that the two-dimensional dynamic problem in question in an anisotropic medium, is reduced to a similar problem in an inhomogeneously-isotropic medium.

2. When the anisotropy of the medium is not complicated by the inhomogeneity, its influence on the dynamic process is described, in general, by a model with curvilinear homogeneous anisotropy ($H = 1$, k_i are constants). It follows from (1.11) that the dynamic process can be studied, in this case, using the analytic functions.

Let us assume that the mesh of the principal directions of the anisotropy can be reduced to an isothermic case. This means that the curvilinear coordinates p_i represent a real and an imaginary part of the analytic function $f(z)$ which connects p_i with x_i. The square of the arc element of the (z)-plane is written in the form

$$dS^2 = c^2 (dp_1^2 + dp_2^2) \tag{2.1}$$

The Lamé coefficients in the isothermic coordinates are $H_1 = H_2 = c$. We note that the non-isothermic mesh p_i (see (1.7)) can be reduced to an isothermic one (see (2.1)) provided that H_i satisfy the conditions $H_i = A_i(p_1) B_i(p_2)$ or $H_1 = A(p_1, p_2) B_i(p_i)$, $i = 1,2$ where A and B are arbitrary functions of the corresponding coordinates.

Under the assumptions made, equations (1.8) can be written in the form

$$cv_i = k_i \partial\varphi / \partial p_i = (-1)^j \partial\psi / \partial p_j \qquad i = 1,\ j = 2; \quad i = 2,\ j = 1 \tag{2.2}$$

In the (z)-plane, the above equations are written in terms of its curvilinear principal directions of anisotropy. Let us introduce the plane of anisotropy $\omega = p_1 + ip_2$ which is connected to the (z)-plane by a conformal transformation. Equations (2.2) describe, in this plane, a dynamic process in a medium with rectilinear anisotropy the principal directions of which coincide with the coordinate axes (see e.g. /5/). The transformation $\xi_1 = \sqrt{k_2/k_1}\, p_1$, $\xi_2 = p_2$, which represents a particular solution of (1.10), reduces the equations (2.2) to the Cauchy–Riemann conditions

$$\partial\varphi_1 / \partial\xi_i = (-1)^j \partial\psi_1 / \partial\xi_j \qquad i = 1,\ j = 2; \ i = 2,\ j = 1,\ \varphi_1 = \varphi,\ \psi_1 = \psi / \sqrt{k_1 k_2} \tag{2.3}$$

3. Using the results of Sect.2, we shall solve a number of specific problems belonging to various areas of the two-dimensional processes in question.

1°. Consider a plane-parallel flow under a mean section of a dam with an impermeable plane sill. We assume that the boundaries of the head and tail bays are continuations of the sill, and we choose the x_1-axis to run along these boundaries. We assume that the horizontal, water-containing stratum, is at the distance d from the x_1-axis, and the principal directions of the soil anisotropy p_i are determined by x_i of the analytic function $\omega = \exp \pi z / (2d)$. The transformation $z = 2d \ln \omega / \pi$ takes the (z)-plane into the anisotropy (ω)-plane. Quasiconformal transformation taking ω to the (ζ)-plane, has the form $\omega = p_1 + ip_2 = \sqrt{k_2 / k_1}\, \xi_1 + i\xi_2$ where k_i denote the soil permeability along the principal directions. In the (ζ)-plane the problem reduces to that of investigating a flow along the ξ_1-axis in homogeneous soil under a dam with a plane sill, with the beginning and the end of the latter determined by the points $\zeta = a_1, b_1$. The vertical direction is impermeable. In the (ζ)-plane the problem is solved with the help of the complex potential /6/

$$w = \frac{\pi(\varphi_B - \varphi_A)}{4K(\sqrt{b_1^2 - a_1^2}/b_1)} \int \frac{d\zeta}{\sqrt{(\zeta^2 - a_1^2)(\zeta^2 - b_1^2)}}$$

where K is a complete elliptic integral of the first kind and φ_B, φ_A denote the constant values of φ along the boundaries of the head and tail bays.

The pressure field associated with $\varphi(x_i)$ and the streamlines $\psi(x_i)$ in the (z)-plane are determined by the complex potential w and the transformation of the (ζ)-plane to the (ω)-and (z)-planes. The size of the sill in the (z)-plane is determined by its end points (a) and (b), and the latter are found by transforming the points $\zeta = a_1, b_1$ in the course of the passage to the (z)-plane.

2°. Let us now consider a problem of emission of heat by a heated rectilinear pipe immersed in an unbounded medium. The heat conductivity of the medium is anisotropic, and the principal directions p_1, p_2 of the anisotropy are situated in the plane $z = x_1 + ix_2 = R \exp \alpha_i$ perpendicular to the axis of the pipe.

Let the principal directions of the anisotropy represent the families of orthogonal parabolas. Then p_i are linked with x_i by the equations

$$x_2^2 = 4p_1^2(p_1^2 - x_1), \quad x_2^2 = 4p_2^2(x_1 + p_2^2)$$

Let us represent the heated pipe in the (z)-plane as a power source Q, and denote the position of the pipe axis by the point z_0. In this case the passage to the anisotropy (ω)-plane is governed by the transformation $z = \omega^2$, and to the (ζ)-plane by the transformation $\zeta = \sqrt{k_2 / k_1}\, p_1 + ip_2$. Taking into account the fact that the passage from (ω) to (z) yields a doubly-sheeted (z)-plane, we can write the solution of the problem in the (ζ)-plane in the form of a complex potential

$$w = (Q / 2\pi) \ln (\zeta^2 - \zeta_1^2)$$

where ζ_1 corresponds to the point z_0.

The temperature distribution $(-\varphi)$ and the streamlines ψ of the thermal flux in the (z)-plane are found from the complex potential w and the transformations of the (ζ)-plane to the (ω) - and (z)-planes.

3°. Consider the problem of distortion in the plane, rectilinear magnetic field strength caused by a circular inclusion of radius R_0. The field has anisotropic magnetic permeability and the principal directions of the anisotropy lie along the non-concentric circles. We write the equation of these circles in the form

$$[x_1 - r^2 a / (r^2 - a^2)]^2 + x_2^2 = [a^2 r / (r^2 - a^2)]^2$$
$$(x_1 - a/2)^2 + (x_2 - a \operatorname{ctg} \theta / 2)^2 = (a / (2 \sin \theta))^2$$

Here the constant a is connected with R_0 by the relation $R_0 = a^2 / (a^2 - 1)$, the center of the inclusion is situated at the point $z = a / (a^2 - 1)$; $a > 1$, and r, θ are parameters determining various curves of the principal directions of the anisotropy of the inclusion.

Let us denote by k the magnetic permeability of the medium outside the inclusion, and by n_i the magnetic permeability of the medium along the principal axes within the inclusion. We pass from the (z)-plane to the anisotropy plane $\omega = r \exp \theta i$ by means of the following conformal transformation:

$$\omega = az / (z - a), \quad z = a\omega / (\omega - a)$$

Quasiconformal transformation of the (ω)-plane to the (ζ)-plane of the form $\zeta = r^{\sqrt{n_1/n_2}} \exp\theta i$ reduces the problem to that of determining, in the (ζ)-plane, a magnetic field due to a dipole at the point $\zeta = a$, in a medium with a continuously homogeneous magnetic permeability k and $\sqrt{n_1 n_2}$ outside and inside the unit circle, respectively. Applying the filtration theorem on a circle /1/, we can write the solution of the problem in the (ζ)-plane in the form

$$w(\zeta_1) = 1/(\zeta_1 - a) + (k - \sqrt{n_1 n_2})/(k + \sqrt{n_1 n_2})(\zeta - 1/a)$$
$$w(\zeta_2) = 2\sqrt{n_1 n_2}/(k + \sqrt{n_1 n_2})(\zeta_2 - a)$$

Here ζ_1 and ζ_2 denote the points outside and inside the circle $|\zeta| = 1$ connected with the anisotropy plane also by the equations $\zeta_1 = \omega$, $\zeta_2 = r^{\sqrt{\mu_1/\mu_2}} \exp \theta i$.

The magnetostatic potential φ and the streamlines of the magnetic field strength vector in the (z)-plane are obtained using the complex potentials $w(\zeta_1)$ and $w(\zeta_2)$, and the transformations of the (ζ)-plane to the (ω)- and (z)-planes.

4°. Consider a plane electric field of constant current in an anisotropic conducting medium. We assume that the field is generated by a charged segment of strength Q, the beginning and end of which have coordinates x_a and x_b. Let the principal directions of the medium anisotropy be situated along ellipses and hyperbolas (with the interfocal distance equal to two), with the specific resistances along these curves equal to k_1 and k_2. The solution of this problem has the form

$$w = (Q \ln \zeta_1) / 2\pi, \quad \zeta_2 = (\zeta_1 + R_0^2 / \zeta_1) / 2R_0$$
$$\zeta_3 = \zeta_2 - A = (\zeta + b^2 / \zeta) / (2b), \quad b = a_0^{\sqrt{k_2 / k_1}}$$
$$\zeta = r^{\sqrt{k_2 / k_1}} \exp \theta i, \quad \omega = r \exp \theta i, \quad z = (\omega + a_0^2 / \omega)/(2a_0)$$

Functions $\varphi(x_i)$ and $\psi(x_i)$ determine the electric potential and current density streamlines; $w(\zeta_1)$ determines the unbounded electric field of a homogeneous plane conductor, generated by a charged, conducting circular boundary. The subsequent transformations represent the solutions of a number of problems obtained according to the scheme of Sect.2, and these solutions clarify the meaning of the constant R_0, A, b and a_0 appearing in the solution as well as their relationship with x_a and x_b.

It should be noted that, in the problems considered, φ and ψ written in terms of the coordinates of the (ω)-plane represent solutions of problems with rectilinear or radial principal directions of anisotropy, and φ, ψ written in terms of the coordinates of the (ζ)-plane are solutions of the problems for homogeneously isotropic media. It should also be noted that the fact that the equations of dynamic processes considered above are all identical, makes it possible to regard every one of the above problems as a solution of the corresponding problem in the other region. Conformal transformations onto the curvilinear surfaces of the (z)-, (ω)- and (ζ)-planes represent solutions of the corresponding problems in curved layers.

In conclusion we note that the equation of the dynamic steady-state process with a nonlinear dynamic law and "ellipsoidal"-type anisotropy, has the form

$$f(v)\, \mathbf{v} = \mathbf{T} \nabla \varphi$$

If the coefficients of the tensor $\mathbf{T}$ are independent of v, then the tensor characterizes the anisotropic and inhomogeneous properties of the medium, independent of the law governing the dynamic process. The function $f(v)$ determines one or another nonlinear law of the process, independent of the properties of the medium. The coefficients of $\mathbf{T}$ dependent of $\mathbf{v}$ characterize the properties of the medium which are dependent on the dynamics of the process, and determine completely its nonlinearity for $f(v) = 1$.

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